A theory for critically divergent fluctuations of dynamical events at non-ergodic transitions

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PACS 05.40.-a –
PACS 02.50.-r –
PACS 64.70.Pf –

Abstract. - We theoretically study divergent fluctuations of dynamical events at non-ergodic transitions. We first focus on the finding that a non-ergodic transition can be described as a saddle connection bifurcation of an order parameter for a time correlation function. Then, following the basic idea of Ginzburg-Landau theory for critical phenomena, we construct a phenomenological framework with which we can determine the critical statistical properties at saddle connection bifurcation points. Employing this framework, we analyze a model by considering the fluctuations of an instanton in space-time configurations of the order parameter. We then obtain the exponents characterizing the divergences of the length scale, the time scale and the amplitude of the fluctuations of the order parameter at the saddle connection bifurcation. The results are to be compared with those of previous studies of the four-point dynamic susceptibility at non-ergodic transitions in glassy systems.

Slow relaxation behavior is observed generally in glassy systems, such as super-cooled liquids, dense colloidal suspensions, foams, emulsions and granular materials. In particular, it has been found that for some systems, the time correlation function exhibits a plateau over a certain time interval. Although the question of whether or not the plateau becomes infinitely long at some transition temperature has not been definitely answered, there is experimental evidence that the plateau does become very long as a particular temperature is approached. This distinctive phenomenon is called a non-ergodic transition [1]. Recently, in addition to studies of the singular behavior of glassy systems, there have been investigations of the growth of the characteristic length scale and the enhancement of the four-point dynamic susceptibility with the goal of elucidating the nature of the cooperative fluctuations of dynamical events responsible for the slow relaxation [2].

As one means for quantifying dynamical events in glassy systems, the following fluctuating quantity has been considered [3, 4]:

$$q_a(\boldsymbol{x},t) = \delta\rho(\boldsymbol{x},t)\int d^3\boldsymbol{y}\, w_a(\boldsymbol{x}-\boldsymbol{y})\delta\rho(\boldsymbol{y},0), \tag{1}$$

where $\delta\rho$ represents the deviation of the local density from its ensemble average and $w_a(\boldsymbol{r}) = \exp(-|\boldsymbol{r}|^2/(2a^2))$ is an overlap function with size a. If $q_a(\boldsymbol{x},t)$ differs significantly from $q_a(\boldsymbol{x},0)$, it can be concluded that a dynamical event leading to the de-correlation of the local density occurs during the time interval $[0,t]$. The ensemble average $\langle q_a(\boldsymbol{x},t)\rangle$ is the time-correlation function of the local density field. For the system with the spatially translational symmetry, it does not depend on $\boldsymbol{x}$. Using the quantity q_a, fluctuations of dynamical events are naturally quantified by the function $C_4(\boldsymbol{r},t) \equiv \langle q_a(\boldsymbol{r},t)q_a(\boldsymbol{0},t)\rangle - \langle q_a(\boldsymbol{0},t)\rangle^2$. Then, it is natural to define the amplitude of fluctuations as $\chi_4(t) \equiv \int d^3\boldsymbol{r} C_4(\boldsymbol{r},t)$. It has been conjectured that $\chi_4(t)$ has a peak at some time $t=t_* \sim |T-T_{\rm c}|^{-\zeta}$, that $\chi_4(t_*)$ diverges as $|T-T_{\rm c}|^{-\gamma}$, and that the correlation length r_* of $C_4(\boldsymbol{r},t_*)$ diverges as $|T-T_{\rm c}|^{-\nu}$, where $T_{\rm c}$ is the temperature at which a non-ergodic transition occurs [4].

There are several theoretical approaches to this phenomenon. A reliable one might be mode coupling theory, because it has been recognized that the theory can provide accurate calculation for some experimental results of glassy systems. Recently, within a framework of mode coupling theory, the exponents of the divergences are calculated [5, 6]. In contrast, as an alternative approach, one seeks for another framework that is independent of specific properties of glassy systems. For example, a space-time thermodynamic formalism has been proposed with this motivation [7]. In this Letter, we present a new theory in the latter approach. The framework we propose is analogous to the Ginzburg-Landau theory of critical phenomena [8]. Within this framework, we calculate the exponents of the divergences by applying a singular perturbation method to a path integral expression.

Framework:. – Let us begin by recalling that the effective Hamiltonian in Ginzburg-Landau theory describes a pitchfork bifurcation in the simplest form; that is, the Ginzburg-Landau potential possesses a double well form

below a critical temperature. This simplicity is responsible for the universality of the results of the Ginzburg-Landau theory. With this property of Ginzburg-Landau theory in mind, the first problem we consider here is to identify the type of the bifurcation occurring at non-ergodic transitions. In fact, this problem is solved in Ref. [9]: The behavior of the time correlation function is determined by an equation, $\partial_t^2\phi = f_\epsilon(\phi, \partial_t\phi)$ that exhibits a saddle connection bifurcation [10]. Here, the order parameter ϕ is defined as follows. First, the equation for the time correlation function $C(\boldsymbol{r}, t)(\equiv \langle\delta\rho(\boldsymbol{r}, t)\delta\rho(\mathbf{0}, 0)\rangle)$ is derived as $\partial_t^2 C = \mathcal{F}(C)$ from a many-body Langevin equation, under the assumption that third- and fourth-order cumulants of density fluctuations are ignored in $\partial_t^2 C$. We then find that the linear operator $\delta\mathcal{F}/\delta C$ around $C = 0$ has a zero eigenvalue at some temperature $T = T_0$ with decreasing T. Using the zero eigenvector $\Phi_{00}(\boldsymbol{r})$ that satisfies $\delta\mathcal{F}/\delta C \cdot \Phi_{00} = 0$ at $T = T_0$, we define the order parameter $\phi(t)$ as the amplitude of $\Phi_{00}(\boldsymbol{r})$ for $C(\boldsymbol{r}, t)$. See Ref. [9] for the detail.

Here, we note that the results presented in this work are independent of the form of f, as long as it exhibits a saddle connection bifurcation at $\epsilon = 0$ with decreasing ϵ, as we find below. For this reason, instead of employing the complicated form of $f_\epsilon(\phi, \partial_t\phi)$ derived for a specific model of many-body Langevin systems in Ref. [9], we assume the simple form $f_\epsilon(\phi, \partial_t\phi) = -\partial_\phi U_\epsilon(\phi)$ with $U_\epsilon(\phi) = -\phi^2\left[(\phi-1)^2 + \epsilon\right]$, where $\epsilon \geq 0$ [11]. The existence of the saddle connection bifurcation in this model is easily seen from the energy conservation condition $(d\phi/dt)^2/2 + U_\epsilon(\phi) = 0$ for trajectories of $\phi(t)$ that satisfy a boundary condition $\phi(\infty) = 0$. This equation exhibits the following behavior. When ϵ is small, ϕ first approaches $\phi = 1$ from $\phi(0) = \phi_0 > 1$, and then after a delay, it begins to approach the origin, $\phi = 0$. As $\epsilon \to 0$, the time interval during which ϕ remains near 1 diverges. This type of bifurcation is termed a saddle connection bifurcation, because the two fixed points $\phi = 1$ and $\phi = 0$, which form saddles in the two-dimensional phase space $(\phi, \partial_t\phi)$, are connected by a trajectory for the case $\epsilon = 0$. There is a clear qualitative correspondence between $\phi(t)$ $(0 \leq t \leq \infty)$ and a time correlation function exhibiting a non-ergodic transition.

Now, we employ the basic idea of Ginzburg-Landau theory. First, in order to describe large-scale spatial variation of the order parameter $\phi(t)$, we replace the order parameter $\phi(t)$ with an order parameter field $\phi(\boldsymbol{x}, t)$ [8]. When fluctuations of ϕ are ignored, as the simplest description of spatially large-scale behavior of the order parameter field, we assume that $\phi(\boldsymbol{x}, t)$ obeys the evolution equation $\partial_t^2\phi = f_\epsilon(\phi) + D\Delta^2\phi$ under the boundary condition $\phi(\boldsymbol{x}, \infty) = 0$, where $D\Delta^2\phi$ represents the diffusive coupling associated with $\partial_t^2\phi$, and we write $f_\epsilon(\phi, \partial_t\phi)$ as $f_\epsilon(\phi)$, for simplicity. Here, $\Delta^2 = (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})^2$. Note also that periodic boundary conditions are first assumed for the system with a size L, and then the limit $L \to \infty$ is taken. Next, we take account of the fluctuation effects of $\phi(\boldsymbol{x}, t)$ by considering space-time configurations of $\phi(\boldsymbol{x}, t)$ with ($\boldsymbol{x} \in \mathbf{R}^3$ and $0 \leq t \leq \infty$), which we denote by $[\phi]$. Then, let $P([\phi])$ be the probability measure for $[\phi]$. Until now, we have not derived the accurate form of $P([\phi])$ on the basis of microscopic models. Nevertheless, relying on a phenomenological argument, we can reasonably assume a form of $P([\phi])$ in the following manner.

First, we divide time t as $t_k = k\Delta t$, with $k = 1, 2\cdots$. Denoting a collection of space configurations of $(\phi(\boldsymbol{x}, t), \partial_t\phi(\boldsymbol{x}, t))$ at an instant time t as $u(t)$, we assume that there is a time interval Δt such that $\tau_{\rm m} \ll \Delta t \ll \tau_{\rm M}$, where $\tau_{\rm m}$ is the longest time scale of fast variables eliminated in deriving the order parameter equation from microscopic models, and $\tau_{\rm M}$ is the characteristic time scale of the time evolution of u. We then consider the transition probability $T(u \to u')$ during a time interval Δt. Now, as in the case of the Onsager-Machlup theory for stochastic processes describing the time evolution of thermodynamic variables [12], the assumption mentioned above suggests the following two properties: (i) $T(u \to u')$ can define a Markov process and (ii) $T(u \to u')$ possesses the large deviation property with respect to time. From the first property (i), we can express $P([\phi])$ as

$$P([\phi]) \simeq P_0(u_0)T(u_0 \to u_1)T(u_1 \to u_2)\cdots, \qquad (2)$$

in the discretized description, where $u_k = u(k\Delta t)$ and $P_0(u_0)$ is the stationary distribution function of u at an instant time. Further, from the second property (ii), T can be expressed as $T(u \to u') \simeq \exp(-\Delta t I(u'|u))$ with a large deviation functional $I(u'|u)$. Here, $I(u'|u) = 0$ should provide the discretized form of the deterministic equation $\partial_t^2\phi = f_\epsilon(\phi) + D\Delta^2\phi$. Thus, approximating $I(u'|u)$ with the second order polynomial of u', we assume

$$I(u'|u) = \frac{1}{2B}\int d^3\boldsymbol{x}$$
$$[(\partial_t\phi)' - \partial_t\phi - \Delta t(f_\epsilon(\phi) + D\Delta^2\phi)]^2/(\Delta t)^2, \quad (3)$$

where B is a constant. From (2) and (3), taking the limit $\Delta t/\tau_M \to 0$, we can express $P([\phi])$ as

$$P([\phi]) = \frac{1}{Z}\exp\left[-\frac{1}{B}F([\phi])\right] \qquad (4)$$

with

$$F([\phi]) = \frac{1}{2}\int_0^\infty dt \int d^3\boldsymbol{x}(\partial_t^2\phi - f_\epsilon(\phi) - D\Delta^2\phi)^2 \qquad (5)$$

under the boundary condition $\phi(\boldsymbol{x}, \infty) = 0$. Here, instead of considering $P_0(u_0)$, we fix the initial condition $\phi(\boldsymbol{x}, 0)$ as ϕ_0 that is determined from the equal-time correlation function. Note that the so-called Jacobian term is not written in (5), because it is independent of $[\phi]$ in the present problem [13].

Although the model (4) with (5) is not justified by the analysis of microscopic models, we expect that this model

can capture essential aspects of fluctuations near the saddle connection bifurcation. With this expectation, we calculate the quantity $\langle \phi(\boldsymbol{x},t)\rangle$ and

$$C_\phi(\boldsymbol{x},t) \equiv \langle \phi(\boldsymbol{x},t)\phi(\boldsymbol{0},t)\rangle - \langle \phi(\boldsymbol{x},t)\rangle \langle \phi(\boldsymbol{0},t)\rangle \quad (6)$$

employing (4), and then from this we compute $\chi_\phi(t) \equiv \int d^3\boldsymbol{x} C_\phi(\boldsymbol{x},t)$. Because $\langle \phi(\boldsymbol{x},t)\rangle$ determines the behavior of $\langle q_a(\boldsymbol{x},t)\rangle$, we believe that $\phi(\boldsymbol{x},t)$ characterizes dynamical events responsible for slow relaxation. We thus conjecture that $\chi_\phi(t)$ has a peak at some $t=t_*$ and that the system displays divergences of the forms $t_* \simeq \epsilon^{-\zeta}$, $\chi_\phi(t_*) \simeq \epsilon^{-\gamma}$, and $r_* \simeq \epsilon^{-\nu}$, where r_* is the correlation length defined from $C_\phi(\boldsymbol{x},t_*)$. In the following, we demonstrate that indeed such divergences do occur and that we have the values $\zeta=2$, $\gamma=3/4$ and $\nu=1/4$.

Analysis:. – The probability distribution (4) can be realized as the stationary distribution of the four dimensional field $\phi(\boldsymbol{x},t)$ that obeys a fictitious stochastic process. Introducing a fictitious time s, we can construct such a stochastic model as

$$\partial_s \phi(\boldsymbol{x},t;s) = -\frac{\delta F}{\delta\phi(\boldsymbol{x},t;s)} + \xi(\boldsymbol{x},t;s), \quad (7)$$

where

$$\langle \xi(\boldsymbol{x},t;s)\xi(\boldsymbol{x}',t';s')\rangle = 2B\delta^3(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')\delta(s-s'). \quad (8)$$

It is easily checked that the s-stationary distribution function of $\phi(\boldsymbol{x},t)$ for this model is exactly expressed by (4). Then, using the operator $\hat{L}_\epsilon(\phi) \equiv \partial_t^2 - \partial_\phi f_\epsilon(\phi)$, we write (7) explicitly as

$$\partial_s\phi = -(\hat{L}_\epsilon - D\Delta^2)(\partial_t^2\phi - f_\epsilon(\phi) - D\Delta^2\phi) + \xi. \quad (9)$$

Now, let $\phi_{\rm cl}(t;\epsilon)$ be the solution of $\partial_t^2\phi = f_\epsilon(\phi)$ with the boundary conditions $\phi_{\rm cl}(0;\epsilon)=\phi_0$ and $\phi_{\rm cl}(\infty;\epsilon)=0$. Then, if $B=0$, all solutions of (9) approach this solution in the limit $s\to\infty$. Furthermore, let $\phi_*(t)$ and $\phi_{\rm B}(t)$ be special solutions of $\partial_t^2\phi = f_0(\phi)$ satisfying the conditions $\phi_*(-\infty)=1$, $\phi_*(\infty)=0$, $\phi_{\rm B}(0)=\phi_0$, and $\phi_{\rm B}(\infty)=1$. Such solutions exist generally at saddle connection bifurcation points. In the model under investigation, we have $\phi_*(t) = (1-\tanh(t/\sqrt{2}))/2$, which represents a kink solution in the t direction, a kind of instanton [14]. Note also that $\phi_{\rm B}(t) \simeq \delta {\rm e}^{-\lambda t}+1$ for $\lambda t \gg 1$ with the constants λ and δ determined by the equation $\partial_t^2\phi = f_0(\phi)$. Thus, for small ϵ, the solution $\phi_{\rm cl}(t;\epsilon)$ is close to $\phi_*(t-\theta_0)+\phi_{\rm B}(t)-1$ with a constant θ_0. Based on these preliminary considerations, we find that, when ϵ and B are small, it is convenient to express solutions of (9) as

$$\phi(\boldsymbol{x},t;s) = \phi_*(t-\theta(\boldsymbol{x},s)) + \phi_{\rm B}(t) - 1 + \varphi(\boldsymbol{x},t;s), \quad (10)$$

where $\theta(\boldsymbol{x},s)$ represents the kink position, which depends on $(\boldsymbol{x},s)$, and $|\varphi|$ is assumed to be small when $\theta\lambda \gg 1$ and $\epsilon \ll 1$. We then find that perturbations of the instanton decay exponentially, except for the Goldstone mode, corresponding to the translation of the instanton in the t direction [15]. It is known that in such a situation a singular perturbation method is useful to derive an evolution equation for the kink position $\theta(\boldsymbol{x},s)$ under the influence of perturbations taking the forms of finite epsilon effects, $\phi_{\rm B}(t)-1$ (see (10)), and noise [16].

This perturbative calculation is carried out by substituting (10) into (9) and extracting the important terms under the conditions $\epsilon \ll 1$, $B \ll 1$, $\theta\lambda \gg 1$ and that spatial variation of θ is small. In order to make these assumptions explicit, we replace $\phi_{\rm B}-1$, ϵ, ∇, and ξ with $\mu(\phi_{\rm B}-1)$, $\mu\epsilon$, $\mu^{1/4}\nabla$ and $\mu^2\xi$ and consider the expansions $\partial_s\theta = \mu\Omega_1 + \mu^2\Omega_2 + \cdots$ and $\varphi = \mu\varphi_1 + \mu^2\varphi_2 + \cdots$, where μ is a small parameter, and Ω_i and φ_i are functions of the field $\theta(\boldsymbol{x},s)$. These replacements are determined in an essentially unique manner by the requirement that a systematic perturbative expansion can be carried out. Then, collecting the terms proportional to μ, we obtain

$$\Omega_1\partial_t\phi_* - \hat{L}_{0*}^2(\phi_{\rm B}-1) = \hat{L}_{0*}[\hat{L}_{0*}\varphi_1 - \epsilon f_{\rm p}(\phi_*) - D\Delta^2\phi_*], \quad (11)$$

where $\hat{L}_{0*} = \hat{L}_0(\phi_*)$ and $f_{\rm p}(\phi) = df_\epsilon(\phi)/d\epsilon|_{\epsilon=0}$. Because $\hat{L}_0\partial_t\phi_* = 0$, the solvability condition for (11) leads to $\Omega_1 = 0$ [17]. Employing this condition, we obtain φ_1. In a similar manner, the solvability condition for the equation consisting of terms proportional to μ^2 yields Ω_2, from which we derive

$$c_1\frac{\partial\theta}{\partial s} = \epsilon c_2 e^{-\lambda\theta} - \epsilon^2 c_3 - \epsilon c_4 D\Delta^2\theta - c_1 D^2\Delta^4\theta + \Xi, \quad (12)$$

where we have ignored terms non-linear in the gradient of θ, such as $(\Delta\theta)^2$ and ${\rm e}^{-\lambda\theta}\Delta\theta$, and Ξ satisfies

$$\langle \Xi(\boldsymbol{x},s)\Xi(\boldsymbol{x}',s')\rangle = 2c_1 B\delta^3(\boldsymbol{x}-\boldsymbol{x}')\delta(s-s'). \quad (13)$$

Note that the coefficients c_i are positive constants independent of ϵ and can be calculated from (9). The first term on the right-hand side of (12) represents the interaction with the exponential tail of the solution $\phi_{\rm B}(t)$. The second term in (12) determines the velocity of the steady propagation of the kink in a system without a boundary in the case $\epsilon \neq 0$.

Now, we investigate the s-stationary statistical properties of θ determined by (12) with (13). We first consider the case $D=0$, in which the field θ behaves independently at each $\boldsymbol{x}$. Thus, according to the central limit theorem, the spatial average of θ over a domain $\Omega \subset \mathbf{R}^3$ obeys a Gaussian distribution with average θ_* and variance $\chi_\theta/|\Omega|$ if the volume of the domain $|\Omega|$ is sufficiently large. In this case, the leading terms of θ_* and χ_θ in the limit $\epsilon\to 0$ can be calculated in an elementary manner and we obtain

$$\theta_* = \frac{B}{c_3\epsilon^2}, \quad (14)$$

$$\chi_\theta = \frac{B^2}{c_3^2\epsilon^4}. \quad (15)$$

Next, we study the case in which D is a small positive number. Specifically, defining

$$\tilde{\theta}_1(\boldsymbol{k}, s) \equiv \int d^3\boldsymbol{x} e^{i\boldsymbol{k}\boldsymbol{x}}(\theta(\boldsymbol{x}, s) - \theta_*), \tag{16}$$

we consider the s-stationary distribution function

$$P_\theta(\{\tilde{\theta}_1\}) = \frac{1}{\tilde{Z}_\theta} \mathrm{e}^{-\frac{\tilde{V}(\{\tilde{\theta}_1\})}{B}}. \tag{17}$$

Although it is difficult to derive the exact form of $\tilde{V}$, we conjecture that $\tilde{\theta}(\boldsymbol{k})$ with small $|\boldsymbol{k}|$ obeys a Gaussian distribution. Then, the fluctuation intensity of $\tilde{\theta}(\boldsymbol{k} = \boldsymbol{0})$ should be equal to χ_θ. Also, considering (12), we find that the length scale of θ is simply determined by the balance among the second, third and fourth terms in the right-hand side when we focus on the regime $\theta\lambda \gg 1$. This leads to the scaling relation $|\boldsymbol{k}|^4 \sim \epsilon$. On the basis of these two considerations, we assume the simple form

$$\tilde{V}(\{\tilde{\theta}_1\}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \left[\frac{B}{2\chi_\theta} |\tilde{\theta}_1(\boldsymbol{k})|^2 + \frac{D_4}{2} \epsilon^3 k^4 |\tilde{\theta}_1(\boldsymbol{k})|^2 \right], \tag{18}$$

where D_4 is a constant. Strictly speaking, (18) is valid only for small $|\boldsymbol{k}|$, but we believe that it provides a good description of the large scale behavior in which we are interested. We therefore use this expression in order to evaluate $\langle \phi(\boldsymbol{x}, t) \rangle$ and $\langle \phi(\boldsymbol{x}, t)\phi(\boldsymbol{x}', t') \rangle$ through the relation (10).

We calculate these quantities by first expressing $\phi_*(t)$ as

$$\phi_*(t) = \int \frac{dz}{2\pi} \tilde{\phi}_* \mathrm{e}^{izt}. \tag{19}$$

Then, using $\langle \phi(\boldsymbol{x}, t) \rangle \simeq \langle \phi_*(t - \theta(\boldsymbol{x})) \rangle$, we write

$$\langle \phi(\boldsymbol{x}, t) \rangle \simeq \int \frac{dz}{2\pi} \tilde{\phi}_*(z) \mathrm{e}^{iz(t-\theta_*)} \left\langle \mathrm{e}^{-iz\theta_1(\boldsymbol{x})} \right\rangle. \tag{20}$$

From the Gaussian nature of the distribution function (17) with (18), we obtain

$$\left\langle \mathrm{e}^{-iz\theta_1(\boldsymbol{x})} \right\rangle = \mathrm{e}^{-\tau_\mathrm{w}^2 z^2/2}, \tag{21}$$

with $\tau_\mathrm{w} = c_5 \epsilon^{-13/8}$, where c_5 is independent of ϵ. Thus, $\langle \phi(\boldsymbol{x}, t) \rangle$ can be expressed as a scaling form $g((t-\theta_*)/\tau_\mathrm{w})$. Also, it is seen that $\langle \phi(\boldsymbol{x}, t) \rangle$ possesses a kink in the t direction whose average position and width are $\theta_* \simeq O(\epsilon^{-2})$ and $\tau_\mathrm{w} \simeq O(\epsilon^{-13/8})$, respectively.

Next, we calculate $\langle \phi(\boldsymbol{x}, t)\phi(\boldsymbol{x}', t) \rangle$ by approximating it with $\langle \phi_*(t - \theta(\boldsymbol{x}))\phi_*(t - \theta(\boldsymbol{x}')) \rangle$. Then, using (19), we write

$$\begin{aligned} \langle \phi(\boldsymbol{x}, t)\phi(\boldsymbol{x}', t) \rangle \quad &\simeq \int \frac{dz}{2\pi} \int \frac{dz'}{2\pi} \tilde{\phi}_*(z)\tilde{\phi}_*(z') \mathrm{e}^{izt+iz't} \\ &\left\langle \exp\left[i \int d^3\boldsymbol{x}'' J^\mathrm{s}(\boldsymbol{x}'')\theta(\boldsymbol{x}'') \right] \right\rangle, \end{aligned} \tag{22}$$

with $J^\mathrm{s}(\boldsymbol{x}'') = -z\delta^3(\boldsymbol{x}'' - \boldsymbol{x}) - z'\delta^3(\boldsymbol{x}'' - \boldsymbol{x}')$. Performing the Gaussian integration of θ, we obtain

$$\begin{aligned} \langle \phi(\boldsymbol{x}, t)\phi(\boldsymbol{x}', t) \rangle \simeq & \int \frac{dz}{2\pi} \int \frac{dz'}{2\pi} \tilde{\phi}_*(z)\tilde{\phi}_*(z') \\ & \mathrm{e}^{iz(t-\theta_*)+iz'(t-\theta_*)} \left\langle \mathrm{e}^{-iz\theta_1(\boldsymbol{x})} \right\rangle \left\langle \mathrm{e}^{-iz'\theta_1(\boldsymbol{x}')} \right\rangle \\ & \exp\left[-zz'\tau_\mathrm{w}^2 G(c_6|\boldsymbol{x} - \boldsymbol{x}'|\epsilon^{1/4}) \right], \end{aligned} \tag{23}$$

where c_6 is independent of ϵ and $G(r) = \mathrm{e}^{-r}\sin(r)/r$. Specifically, in the long distance regime, for which we have $|G(c_6|\boldsymbol{x} - \boldsymbol{x}'|\epsilon^{1/4})| \ll 1$, we can derive the approximate expression

$$C_\phi(\boldsymbol{x}, t) \simeq \tau_\mathrm{w}^2 \left[\partial_t \langle \phi(\boldsymbol{0}, t) \rangle \right]^2 G(c_6|\boldsymbol{x}|\epsilon^{1/4}). \tag{24}$$

From this expression (23), we immediately find that the correlation length is proportional to $\epsilon^{-1/4}$. Thus, we obtain $\nu = 1/4$. Next, noting that the singular part of $\chi_\phi(t)$ originates from the long-distance part of $C_\phi(\boldsymbol{x}, t)$, we can use the asymptotic form (24) in order to evaluate the exponents ζ and γ. First, it is found that $\chi_\phi(t)$ has a peak at $t = \theta_*$, and thus, from (14), we obtain $\zeta = 2$. Furthermore, from the relation $\tau_\mathrm{w}\partial_t g((t - \theta_*)/\tau_\mathrm{w}) \sim O(\epsilon^0)$, we obtain $\chi_\phi(\theta_*) \simeq \epsilon^{-3/4}$, and hence $\gamma = 3/4$. Finally, if the asymptotic formula (24) is valid on the correlation length scale, it suggests that the critical exponent η in the standard notation of critical phenomena is equal to -1.

Conclusion:. – We have presented a phenomenological framework for determining the statistical properties of dynamical events at non-ergodic transitions. Within this framework, we have obtained the results $\zeta = 2$, $\gamma = 3/4$ and $\nu = 1/4$, which are to be compared with the values obtained in numerical experiments ($\gamma/\zeta \simeq 0.4$ and $\nu/\zeta \simeq 0.22$) [18] and the values calculated using the mode coupling theory ($\zeta \simeq 2.3$ for a Lennard-Jones fluid, $\gamma = 1$ [5,6], and $\nu = 1/2$ [5] and $\nu = 1/4$ [6].) [19]. It might be rather surprising that our theoretical values are close to the results of the mode coupling theory despite the great difference between the two frameworks. However, because both theories are still at the mean field level in the sense that critical fluctuations are not treated precisely, it is not time to compare the results in detail. Similarly, with regarding the discrepancy with the results by numerical simulations, we wish to be cautious of a hasty judgment. Rather, the important result specific to our theory is that the divergence of the fluctuation is caused by the Goldstone mode for the instanton. We expect that this mechanism should be taken into account by combining a singular perturbation method with a path integral approach for the analysis of microscopic models.

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The authors thank K. Kawasaki, K. Miyazaki and H. Tasaki for useful comments on the manuscript. This work was supported by a grant from the Ministry of Education, Science, Sports and Culture of Japan (No. 16540337).

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